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On minimal imperfect graphs without induced P_5

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Abstract

In this paper, we summarize previous known results on P_5-free minimal imperfect graphs (i.e. minimal imperfect graphs not containing a path on 5 vertices as induced subgraph) and we introduce two new classes of graphs (defined by a local property) that contain P_5-free graphs. Next, we show that most of the results concerning P_5-free graphs can be extended to these classes. Moreover, we present a structural characterization of these graphs which leads to some new results. In particular, we prove that the Strong Perfect Graph Conjecture holds true for P_5-free and F-free graphs where F is any connected configuration on 5 vertices not containing an induced $2K_2$. © 1999 Elsevier Science B.V. All rights reserved.

1. Introduction

A graph is *perfect* if the vertices of any induced subgraph H can be colored, in such a way that no two adjacent vertices receive the same color, with a number of colors (denoted by $\chi(H)$) not exceeding the cardinality $\omega(H)$ of a maximum clique of H.

A graph is *minimal imperfect* if all its proper induced subgraphs are perfect but it is not. In particular, $\omega(G)+1=\chi(G)$ for G minimal imperfect. All the notions not defined here may be found in [4].

It is an easy task to check that an odd chordless cycle of length at least five (usually called a *hole*), as well as its complement (usually called an *anti-hole*) are minimal imperfect graphs.

This remark and some early results concerning perfect graphs determined Berge [3] to formulate the following two conjectures (known as the Strong and the Weak Perfect Graph Conjecture)

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(SPGC) A graph is perfect if and only if it does not contain an odd hole or an odd anti-hole as an induced subgraph.

(WPGC) A graph is perfect if and only if its complement is perfect.

A classical way to approach SPGC is to consider classes of graphs defined by forbidden configurations. In particular, if F is a four-vertex graph, the conjecture has been shown to hold true for F-free graphs in almost all cases, except when F is either a cycle on four vertices (C_4) or its complement ($2K_2$). In this paper, we are interested in the class of P_5-free graphs, which is larger than $2K_2$-free graphs, and we will mainly show:

Theorem 1. *SPGC holds for the (P_5,F)-free graphs whenever F is any connected configuration on five vertices not containing an induced $2K_2$.*

We recall that there exists exactly 21 connected graphs on 5 vertices; those graphs are displayed in Fig. 1. Note that the only two configurations containing a $2K_2$ are F_1 and F_2, and remark that proving SPGC for one of them would prove that a P_5-free minimal imperfect Berge graph must contain a $2K_2$ and therefore that SPGC is true for $2K_2$-free graphs.

First note that SPGC is already known to hold for several classes of F-free graphs: when F is a *bull* (an F_{13}), see [11], a *dart* (an F_9) see [36], a *chair* (an H_1) and a *co-chair* (an F_{10}) see [32].

Moreover, SPGC is also known to hold for certain classes of (P_5,F)-free graphs: when F is an F_5, an F_6, or a $\overline{P_5}$ this follows from a result due to Hayward [17] on *Murky graphs* (i.e. on graphs that contain no C_5, no P_6 and no $\overline{P_6}$), when F is an F_{12} this follows from a result due to Olariu [29] on *Pan-free graphs* (i.e. on graphs that contain no induced subgraph isomorphic to a k-pan, $k \geqslant 4$, where a k-pan is composed from a chordless cycle C_k on k vertices and a vertex a outside C_k which is adjacent to exactly one vertex of C_k), while Maffray and Preissmann [21] have shown that SPGC holds for (P_5,K_5)-free graphs.

Thus there remains eight F's for which the theorem must be proved. As a matter of fact we will prove some stronger results; for instance we will prove that both $(P_5,K_{2,4})$-free and (P_5,F')-free graphs (where F' is the complete join of P_2+P_1 with a 3-stable set) satisfy SPGC.

2. Minimal imperfect graphs

While the Strong Perfect Graph Conjecture is still unsettled, the Weak Perfect Graph Conjecture is an easy consequence of the following theorem of Lovász [19]:

Theorem 2 (The Perfect Graph Theorem). *A graph $G=(V,E)$ is perfect if and only if for every induced subgraph H of G the following inequality holds:*

$$\omega(H).\alpha(H) \geqslant |H|.$$

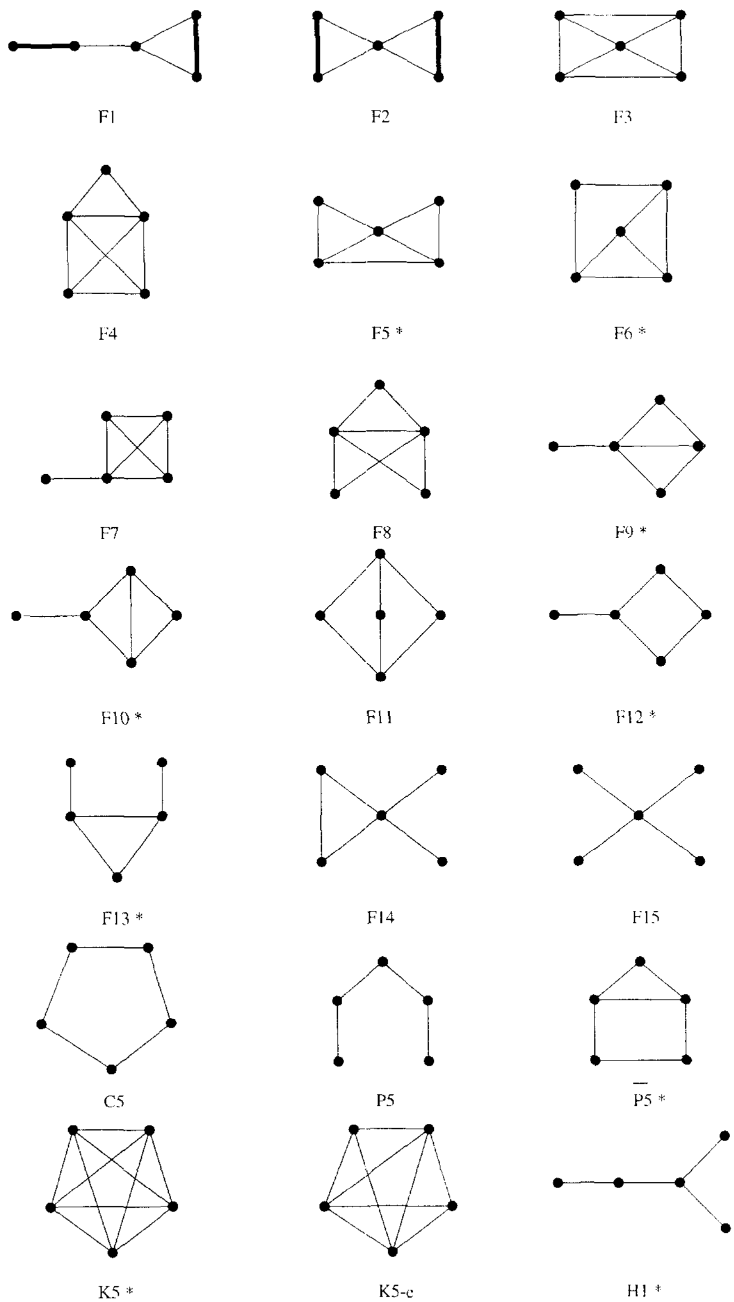

Fig. 1. The 21 connected configurations on 5 vertices (SPGC is known to hold for (P_5, F)-free graphs, where F is any of the configurations labeled with a star).

One can deduce from this theorem [19,31], that in a minimal imperfect graph G, ($n=|V(G)|$),

(1) $n=\alpha.\omega+1$,

(2) for every vertex $v\in V(G)$ $G-v$ has a unique partition into α ω-cliques (i.e. a clique of size ω) and a unique partition into ω α-stable sets.

(3) each vertex of G is in exactly α α-stable sets and in exactly ω ω-cliques.

(4) for every α-stable set S of G, there is a unique ω-clique $Q(S)$ of G such that $S\cap Q(S)=\emptyset$; for every ω-clique Q of G, there is a unique α-stable set $S(Q)$ of G such that $Q\cap S(Q)=\emptyset$.

Bland et al. [7] defined a graph to be *partitionable* if there exist two integers $\alpha,\omega\geqslant 2$ such that (1) and (2) hold. Further refinements along this line are due to Padberg [31]. We only need the following property:

Proposition 3. *In a minimal imperfect graph G, given two vertices u and v, there exists an ω-clique containing u and not containing v.*

Proof. Use property (2) of a minimal imperfect graph. □

Two of the most useful graphical properties of minimal imperfect graphs were found by Meyniel, Bertschi and Reed, Fonlupt and Uhry, and Chvátal. To describe their results we need recall a few definitions. Two nonadjacent vertices x,y form an *even pair* if all chordless paths joining x to y have an even number of edges. A set C of vertices of a graph G is called a *star-cutset* if $G-C$ is disconnected and in C there is a vertex x adjacent to all other vertices of C.

Lemma 4 (Meyniel [25]; Bertschi and Reed [5,6]; Fonlupt and Uhry [13]). *No minimal imperfect graph contains an even pair.*

Lemma 5 (V. Chvátal [8] – Star-cutset lemma). *No minimal imperfect graph contains a star-cutset.*

Moreover, one can extend star-cutset lemma for any partitionable graph (see Maire's Ph.D. Thesis [23] for a proof of this fact).

We denote by $N_G(x)$ the set of vertices of G adjacent to x; when there can be no confusion we shall write $N(x)=N_G(x)$. Lemma 5 implies that if G is minimal imperfect then the graph induced by $V-(v+N(v))$, for every v in V, must be connected. Moreover, Gallai [15] (see also [26,27,30]) has shown that $N_G(v)$ induces a connected subgraph of $\overline{G}$ (otherwise, $\{v\}\cup N_{\overline{G}}(v)$ induces a star cutset in $\overline{G}$). A graph is called *Berge* if it contains no odd hole and no odd antihole.

Let $G=(V,E)$ be a minimal imperfect graph and let u,v be two nonadjacent vertices of G. We denote by $G+uv$ the graph $(V,E\cup\{(u,v)\})$ and one says that u,v is a *co-critical* pair if $\omega(G+uv)=\omega(G)+1$. If uv is an edge of G and $\alpha(G-uv)=\alpha(G)+1$, uv is said to be a *critical* edge.

Theorem 6 (Giles et al. [16]). *No minimal imperfect Berge graph contains a cycle of co-critical pairs.*

Theorem 7 (Bacsó [1] and Sebő [33]). *No minimal imperfect Berge graph contains the following configuration (x, y_1, y_2, z_1, z_2) on five vertices:*

- *xy_1 and xy_2 induce two critical edges*
- *xz_1 and xz_2 are two co-critical pairs.*

Meyniel [24] proved that a graph is perfect if each of its odd cycles with at least five vertices contains at least two chords. Nowadays, these graphs are known as *Meyniel* graphs. We can remark that if G is a Meyniel graph then the complement $\overline{G}$ of G does not contain a P_5 (otherwise G would contains a $\overline{P_5}$). Therefore, the class of P_5-free graphs contains the class of co-Meyniel graphs. Hence, proving SPGC for P_5-free graphs would be a generalization of Meyniel's theorem.

Before studying P_5-free graphs, we shall recall some known results on $2K_2$-free graphs. In [33] Sebő conjectures that every minimal imperfect graph contains a *forcing* (i.e. two ω-cliques with $\omega - 1$ vertices in common) moreover if G contains a forcing then $\overline{G}$ contains a critical edge and therefore, if this conjecture is true, the following theorem implies SPGC for *normalized* $2K_2$-free graphs (i.e. $2K_2$-free graphs such that every edge belongs to some ω-clique).

Theorem 8 (Bacsó [1]). *Let G be a $2K_2$-free minimal imperfect Berge graph, normalized, then G does not contain a critical edge.*

Following De Simone and Gallucio [12], we shall say that two nonadjacent vertices x, y in a graph G are *witnesses* if every other vertex is adjacent to at least one of them. This implies, in particular, that in $\overline{G}$ the edge xy belongs to no triangle. In [12] De Simone and Gallucio prove:

Theorem 9 (De Simone and Gallucio [12]). *SPGC is true iff every minimal imperfect graph or its complement has an edge that belongs to no triangle.*

This theorem provides an alternative way of proving the validity of SPGC for special classes of graphs. Indeed, it suffices to choose a graph F and to show that every minimal imperfect Berge graph that contains F has one of the following properties:

- G has an edge e that belongs to no triangle such that $G - e$ is F-free.
- $\overline{G}$ has an edge e that belongs to no triangle such that $\overline{G} - e$ is $\overline{F}$-free.

Let G be a $2K_2$-free Berge graph, it is easy to see that the graph G' obtained from G by adding an edge between two witnesses is still $2K_2$-free. Hence, to show that $2K_2$-free Berge graphs are perfect, we only need to show that every such graph has two witnesses.

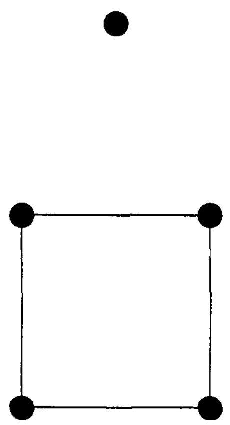

Fig. 2. The graph H.

Theorem 10 (De Simone and Gallucio [12]). *Let G be a $2K_2$-free minimal imperfect Berge graph. If G does not contain the graph H in* Fig. 2, *then it has two witnesses.*

Corollary 11 (De Simone and Gallucio [12]). *Every $2K_2$-free and H-free Berge graph is perfect.*

In [20] Lubiw conjectures that in a minimal imperfect Berge graph the subgraph induced by the neighbourhood of any vertex is connected. In [23, Ch. 8] Maire proves this conjecture for $2K_2$-free graphs.

Theorem 12 (Maire [23]). *Let G be a $2K_2$-free minimal imperfect Berge graph, then the neighbourhood of any vertex is connected.*

We shall see that for P_5-free graphs this conjecture is also true, in fact, we will prove a stronger statement.

3. On IP_5 and MP_5 graphs

Let $G=(V,E)$ be a graph and let $v \in V(G)$. We say that v is *Mid* P_5 if there exist four other vertices a,b,c and d in $V(G)$ such that $abvcd$ induces a P_5 in G. We say that v is *Int* P_5 if there exists a P_5 such that v is an internal vertex of this P_5 (i.e. if we denote this P_5 by $abcde$, then $v \in P_5$ and $v \neq a,e$). A graph is said to be IP_5 (resp. MP_5) if there exists, in G, a vertex v which is not Int P_5 (resp. not Mid P_5); note that we have the following inclusions:

$$P_5\text{-free} \subseteq IP_5 \subseteq MP_5.$$

Theorem 13. *Let G be a C_5-free partitionable graph, then for every vertex $v \in V(G)$, which is not Mid P_5, the subgraph induced by $N_G(v)$ is connected.*

This theorem leads to a structural characterization of $G - v$. Let v be a vertex of G and let $M(v) = M = \{u \in V \mid u \neq v, uv \notin E\}$. We denote by $\mathcal{M}(w)$, for $w \in N(v)$, the subset $M \cap N_G(w)$.

Theorem 14. *Let $G=(V,E)$ be a C_5-free partitionable graph, for every vertex $v \in V$ such that v is not Mid P_5, one has a partition of $V(G)$ in*

$v, W, Y,$ and M

such that $N(v)$ is partitioned in W and Y with $Y \neq \emptyset$ connected and $W = \{w\} + N_{N(v)}(w)$, where w is a vertex such that $M = \mathcal{M}(w)$. □

This theorem can be used to answer a conjecture due to Hoàng [18] in the case of P_5-free graphs.

Conjecture 18 (Hoàng [18]). *Let x and y be two vertices of a minimal imperfect graph G with $\omega(G) \geqslant 3$. Then in $G - xy$ there is a chordless path of length two having x and y as endpoints.*

We can see that this conjecture holds true if x or y are not Mid P_5. Indeed, if $xy \notin E$, then such a path exists using $w \in N(x)$ and if $xy \in E$, since $|N(x)| \geqslant 2$ (because $\omega \geqslant 3$) and since $N(x)$ induces a connected subgraph of G there exists a vertex u such that xuy induces the desired path.

Corollary 16 (Odd pair conjecture on P_5-free graphs). *No P_5-free minimal imperfect Berge graph contains an odd pair (i.e. two vertices u_1, u_2 of a graph G such that all chordless paths of $G - u_1u_2$ have an odd number of edges).*

We can also deduce from this structure the following relations:

Theorem 17. *Let G be a minimal imperfect Berge graph; then for every vertex $v \in V(G)$ such that v is not Mid P_5 one has*
- $d(v) \geqslant \alpha + \omega - 1$,
- $d(v) \geqslant 2(\alpha - 1)$.

Remark 18. Sebő [34] has shown that in a minimal imperfect graph G, for all minimal cutset C of G, one has

$$|C| \geqslant 2(\omega - 1)$$

and has conjectured that in a minimal imperfect graph, there exists a vertex of degree $2\omega - 2$.

We will prove that, for (P_5, C_5)-free graphs, this conjecture is equivalent to SPGC. First, we can remark that in an odd hole or an odd anti-hole, every vertex is of degree $2\omega - 2$. Then, we show that

Theorem 19. *Let G be a C_5-free partitionable graph. If there exists a vertex v which is not Int P_5 and such that $d(v) = 2\omega - 2$ then $G \simeq \overline{C_{2p+1}}$.*

Theorem 14 also gives some informations on the structure of the subgraph induced by the neighbourhood of any vertex.

Theorem 20. *Let G be a C_5-free minimal imperfect graph. If there exists a vertex v which is not Mid P_5 and such that $N_G(v)$ induces a split graph (i.e. a graph whose vertex set can be partitioned into a maximal clique and a stable set), then $G \simeq \overline{C_7}$.*

We know that, in a minimal imperfect graph, the neighbourhood of any vertex induces a minimal cutset. In [9] Chvátal proposes the following conjecture:

Conjecture 21. *Every minimal imperfect Berge graph G has the following properties:*
(1) *For each cutset C of G, the subgraph of G induced by C is connected.*
(2) *For each cutset C of G, the subgraph of G induced by C contains a P_4.*

In [2] we have proved this conjecture for P_5-free graphs,

Theorem 22 (Barré and Fouquet [2]). *Let G be a (P_5, C_5)-free minimal imperfect graph and let C be a minimal cutset of G; then,*
- *C induces a connected subgraph of G,*
- *C contains a P_4.*

The preceding theorem shows that a minimal cutset, in a (P_5, C_5)-free minimal imperfect graph, cannot be P_4-free. We shall see now that such a minimal cutset cannot belong to a particular class of graphs containing P_4-free graphs.

We call *complete join* of two (vertex disjoint) graphs $A = (V_A, E_A)$ and $B = (V_B, E_B)$ the graph with the vertex set $V_A \cup V_B$ and edge set $E_A \cup E_B \cup \{ab \mid a \in V_A, b \in V_B\}$.

Seinsche [35] has shown that P_4-free graphs can be constructed by complete join and disjoint union from isolated vertices. From this characterization, we shall define a new class of graphs; as a matter of fact we shall replace isolated vertices by a new family (called $\mathscr{B}^*$) containing them. Let $\mathscr{B}$ be the family of bipartite graphs (containing isolated vertices) and let $\mathscr{B}^*$ be the family defined by
- $\mathscr{B} \subseteq \mathscr{B}^*$
- $\forall G_1, G_2 \in \mathscr{B}^*$, the complete join and the disjoint union of G_1 and G_2 are in $\mathscr{B}^*$.

Gallai [15] (see also [26,27,30]) proved that, for each vertex v in a minimal imperfect graph G, the neighbourhood of v induces a connected subgraph of the complement of G; hence if $N(v) \in \mathscr{B}^*$ we have $N(v)$ disconnected or $N(v)$ induces a bipartite graph (this last case is impossible if G is Berge since this implies that $\omega(G) = 3$ and Tucker [37] has shown that SPGC is true for K_4-free graphs).

Therefore, if Lubiw's conjecture (in G minimal imperfect and Berge, $\forall v \in V(G)$, $N(v)$ is connected [20]) is true, for each vertex v of G we have $N(v) \notin \mathscr{B}^*$. Remark that if we define $\mathscr{B}_s^*$ as $\mathscr{B}^*$ where $\mathscr{B}_s$ if the family of graphs which are either bipartite or split, for the same reason as before and since $N(v)$ cannot induces a split graph, we have: *Let G be a minimal imperfect Berge graph and let v be a vertex which is not Mid* P_5, *then* $N(v) \notin \mathscr{B}_s^*$.

For P_5-free graphs we have a stronger result:

Theorem 23 (Barré and Fouquet [2]). *Let G be a* P_5*-free minimal imperfect Berge graph and let C be a minimal cutset of G, then* $C \notin \mathscr{B}^*$.

Lastly, to end our review of the properties of P_5-free ($2K_2$-free) minimal imperfect graphs, we have the following two results:

Theorem 24. *Let G be a minimal imperfect* $2K_2$*-free graph and let H be a subgraph of G, of size at most* ω *and hamilton connected. Then there exists a Hamiltonian cycle in G that extends any Hamiltonian path in H.*

Therefore, if we take for H any ω-clique of G, then we can always find a Hamiltonian cycle such that all the vertices from H are consecutive on this cycle. For MP_5 graphs we have:

Theorem 25. *Let G be a* C_5*-free minimal imperfect* MP_5 *graph, let* $v \in V(G)$ *(v not Mid* P_5*) and let* $K_1, K_2, \ldots, K_\alpha$ *be the partition of* $G - v$ *in* $\alpha\omega$*-cliques. Then there exists a Hamiltonian cycle of G such that each* ω*-clique* K_i *has its vertices consecutive on this cycle.*

4. On some (P_5, F)-free graphs

Another way to study P_5-free minimal imperfect graphs is to consider P_5-free graphs that do not contain some configurations on 5 (or more) vertices as induced subgraph. Here, we shall consider all the connected configurations on 5 vertices not containing an induced $2K_2$ and we shall see that SPGC is true for those graphs. We recall that there exists exactly 21 connected graphs on 5 vertices; those graphs are displayed in Fig. 1.

Theorem 26 (De Simone and Gallucio [12]). *Let G be a Berge graph not containing an* H_1 *(called a* chair*) or an* H_2 *(see* Fig. 3*) as induced subgraph, then G is perfect.*

Corollary 27 (Olariu [28] and Hoàng [18]). *A Berge graph is perfect if it contains no* P_5 *and no* H_1.

Theorem 28. *Let G be a Berge graph with no* P_5 *and no* F_{10} *(i.e. no* anti-chair*), then G is perfect.*

In fact, these two results are corollaries of the following more general theorem due to Sassano.

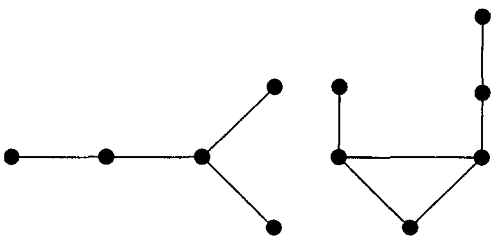

Fig. 3. H_1 and H_2.

Theorem 29 (Sassano [32]). *Chair-free Berge graphs are perfect.*

Applying the same technique as in [18], we obtain

Theorem 30. *A Berge graph is perfect if it contains no P_5 and no F_{12}.*

One can remark that this result can be derived from a theorem due to Olariu; indeed, it is easy to see that a (P_5, F_{12})-free Berge graph is pan-free and Olariu proves

Theorem 31 (Olariu [28]). *SPGC holds true for pan-free graphs.*

Theorem 32. *Let G be a Berge graph with no P_5 and no F_{14}, then G is perfect.*

Some theorems of this type can be derived from the structural characterization of P_5-free minimal imperfect graph (Theorem 14).

Theorem 33. *Every P_5-free and $(K_5 - e)$-free Berge graph is perfect.*

Theorem 34. *Let G be a MP_5 minimal imperfect Berge graph; then G contains a $K_{2,\alpha}$. Moreover every vertex v of G which is not Mid P_5 is in the stable set of size 2 of a $K_{2,\alpha}$.*

Corollary 35 (De Simone and Gallucio [12]). *A Berge graph is perfect if it contains no P_5 and no $K_{2,3}$ (denoted by F_{11}).*

Tucker [37] proved that K_4-free graphs satisfy SPGC, therefore any minimal imperfect Berge graph G is such that $\omega(G) \geqslant 4$. Moreover, by the Perfect Graph Theorem, we can suppose that $\alpha(G) \geqslant 4$ and then

Corollary 36. *A MP_5 Berge graph is perfect if it contains no F_{15}.*

In the case of P_5-free graphs, Maffray and Preissmann have shown that we can suppose $\omega(G) \geqslant 5$; we shall extend their result to IP_5 graphs.

Theorem 37. *Every K_5-free IP_5 Berge graph is perfect.*

Corollary 38 (Maffray and Preissmann [21]). *Every P_5-free and K_5-free Berge graph is perfect.*

One fact in [21] leads to the following theorem.

Theorem 39. *Let G be an IP_5 Berge graph with no F_4, then G is perfect.*

Now, let us recall a theorem due to Hayward. A graph is said to be *Murky* if neither the graph nor its complement contains a C_5 or a P_6.

Theorem 40 (Hayward [17]). *Murky graphs are perfect.*

Since the complement of a P_6 contains an F_5, an F_6 and a $\overline{P_5}$ as induced subgraphs, this theorem implies the perfection of Berge graphs either with no P_5 and no F_5, or with no P_5 and no $\overline{P_5}$ (see [14] for a characterization of such graphs), or with no P_5 and no F_6 (see also Corollary 43).

Theorem 41. *Let G be a P_5-free minimal imperfect Berge graph; then there exists at least one vertex v of G such that $G - v$ contains a subgraph isomorphic to the complete join of a K_2 (i.e. an edge, which is, moreover, included in $M(v)$) and an $(\alpha - 1)$-stable set (and this $(\alpha - 1)$-stable set is included in $N(v)$).*

Corollary 42. *Let G be a Berge graph with no P_5 and no F_8, then G is perfect.*

Corollary 43. *Let G be a P_5-free Berge graph with no induced F_6, then G is perfect.*

In this last case, we have a stronger result.

Theorem 44. *Let G be a MP_5 Berge graph with no induced F_6, then G is perfect.*

Theorem 45. *Let G be a Berge graph with no P_5 and no F_3, then G is perfect.*

Note that we will look into configuration F_7 in the next section, before the proof of Theorem 28. So, we have proved our main theorem. Theorem 1. *SPGC holds for (P_5, F)-free graphs whenever F is any connected configuration on 5 vertices not containing an induced $2K_2$.*

5. The proofs

Let $G=(V,E)$ be a graph and let $v \in V$. For every vertex $w \in N(v)$ we denote by $\mathcal{M}(w)$ the subset of vertices, neighbours of w, which are not in $v+N(v)$ and for a subset Y of V we write $N_Y(u)=N(u)\cap Y$.

Lemma 46. *Let G be a C_5-free graph, let $v \in V$ such that v is not Mid P_5 and let $x,y \in N(v)$ such that $xy \notin E$; then $\mathcal{M}(x)\subseteq\mathcal{M}(y)$ or $\mathcal{M}(y)\subseteq\mathcal{M}(x)$ (if $\mathcal{M}(x)$ and $\mathcal{M}(y)$ are nonempty).*

Proof. Suppose that there exist $x,y \in N(v)$ contradicting the hypothesis. So, there exist two vertices $a \in \mathcal{M}(x)\backslash\mathcal{M}(y)$ and $b \in \mathcal{M}(y)\backslash\mathcal{M}(x)$ such that $\{a,x,v,y,b\}$ induces a P_5 or a C_5, a contradiction. □

Assume that G is a partitionable graph and let $S_1,\ldots,S_\omega$ be the partition of $G-v$ in ω α-stable sets. One can remark that for every vertex x in $N(v)$, one has $\mathcal{M}(x)$ non-empty (since G has no star-cutsets). We write $N_i=N(v)\cap S_i$ and $M_i=S_i\backslash N(v)$.

Lemma 47 (Tucker [31]). *$\forall i,j$ $(1\leqslant i,j\leqslant\omega, i\neq j)$ $G[S_i\cup S_j]$ induces a connected bipartite graph.*

The following lemma, which appear in [21] for P_5-free graphs, can be extended to IP_5 graphs and will be frequently used.

Lemma 48. *Let G be C_5-free and assume that v is not Int P_5. If there exists an edge ab with $a \in M_i$ and $b \in M_j$ $(i\neq j)$, then every vertex of N_i is adjacent to every vertex of N_j.*

Proof. Let G be a C_5-free partitionable graph, let a,b be as in the hypothesis and suppose that $N_i\cup N_j$ does not induce a complete bipartite subgraph. We can remark that $\forall i$, $N_i\neq\emptyset$; otherwise $\{v\}\cup M_i$ would induce an $(\alpha+1)$-stable set. Let xy be a non-edge in $N_i\cup N_j$, by Lemma 47 there exists a shortest path P joining $\{a,b\}$ to $\{x,y\}$, we choose the vertices a and b minimizing the distance between $\{a,b\}$ and $\{x,y\}$ (i.e. minimizing the length of P). Without loss of generality we may assume that one of the extremities of P is a, and then $b\notin P$. Let b' be the vertex that follow a on P, then $b'\in N_j$ otherwise the edge ab' would contradict the choice of $\{a,b\}$.

Case 1. $b'=y$: We know that $xa\notin E$ which implies that $\mathcal{M}(x)\subseteq\mathcal{M}(y)$ (Lemma 46) and, in particular, $xb\notin E$. But then $bayvx$ induces a P_5 containing v as an internal vertex, a contradiction.

Case 2. $b'\neq y$: By the minimality of P, we have $ay\notin E$ and then $bab'vy$ induces a P_5 that contains v as an internal vertex, a contradiction. □

Now, we shall recall some properties of P_5-free connected bipartite graphs. Let B be such a graph on $V = V_1 \cup V_2$ and let $x \in V_1$, we denote by $N_2(x)$ the neighbourhood of x in V_2. The following three properties are equivalent:

(1) $\forall x, x' \in V_1$, $N_2(x) \subseteq N_2(x')$ or $N_2(x') \subseteq N_2(x)$.

(2) B contains no induced P_5.

(3) $\forall y, y' \in V_2$, $N_1(y) \subseteq N_1(y')$ or $N_1(y') \subseteq N_1(y)$.

In particular, this implies that there exists a vertex $v_1 \in V_1$ (resp. $v_2 \in V_2$) such that $N_2(v_1) = V_2$ (resp. $N_1(v_2) = V_1$), the edge v_1v_2 is called a *bi-universal* edge of B.

Proof of Theorem 13. Let G be a C_5-free partitionable graph, let $v \in V(G)$ which is not Mid P_5 and suppose that $N_G(v)$ is not connected. Let X and Y be two connected components of $N(v)$ and let $w \in X \cup Y$ be such that $\mathcal{M}(w)$ is maximal for inclusion (Lemma 46). One can suppose that $w \in X$, then $\forall y \in Y$, $\mathcal{M}(y) \subseteq \mathcal{M}(w)$ and $w + v + \mathcal{M}(w)$ induces a star-cutset as soon as one of the following three properties is satisfied

- $X \neq \{w\}$,
- $M - \mathcal{M}(w) \neq \emptyset$,
- there exists a third connected component Y' in $N(v)$.

So, suppose that none of these properties is satisfied. We claim that no vertex in Y can be adjacent to all the other vertices in Y. Otherwise, let $y \in Y$ be such a vertex, if $\omega \geqslant 3$ any ω-clique containing v must intersect Y and thus must contain y, which contradicts Property 3, and if $\omega = 2$ it is not difficult to check that $G \simeq C_5$, a contradiction.

Now, let $y \in Y$ such that $\mathcal{M}(y)$ is maximal for inclusion and let $Y' = Y \setminus (\{y\} + N_Y(y))$, then $Y' \neq \emptyset$ and $\forall y' \in Y'$, $\mathcal{M}(y') \subseteq \mathcal{M}(y)$. Therefore $y + v + N_Y(y) + \mathcal{M}(y)$ induces a star disconnecting w and Y', a contradiction. □

Proof of Theorem 14. Since G is a C_5-free partitionable graph and $v \in V(G)$ is not Mid P_5, we know that $N(v)$ is connected (Theorem 13). Let w be a vertex of $N(v)$ such that $\mathcal{M}(w)$ is maximal for inclusion and let $Y = N(v) \setminus (w + N_{N(v)}(w))$. We have $Y \neq \emptyset$, otherwise any ω-clique containing v also contains w, which is impossible. Now, we shall show that $M = \mathcal{M}(w)$. We know that $\forall y \in Y$ $\mathcal{M}(y) \subseteq \mathcal{M}(w)$ (Lemma 46), so if $M \setminus \mathcal{M}(w) \neq \emptyset$, the set $w + N_{N(v)}(w) + \mathcal{M}(w)$ induces a star disconnecting $M \setminus \mathcal{M}(w)$ and Y. Lastly, if Y is not connected then $w + v + N_{N(v)}(w) + \mathcal{M}(w)$ induces a star-cutset, a contradiction. □

Proof of Theorem 17. Let G be a minimal imperfect Berge graph and let $v \in V(G)$ (which is not Mid P_5). We define $w \in N(v)$ and $M = \mathcal{M}(w)$ as in Theorem 14.

1. Let $S_1, S_2, \ldots, S_\omega$ be the partition of $G - v$ in ω α-stable sets. We can suppose that $w \in S_1$ and then $S_1 \subseteq N(v)$, moreover $\forall i, 2 \leqslant i \leqslant \omega$ $|S_i \cap N(v)| \geqslant 1$ (otherwise $\{v\} \cup S_i$ would induce an $(\alpha + 1)$-stable set). So,

$$\forall v \in V(G), \quad d(v) = |S_1| + \sum_{i=2}^{\omega} |S_i \cap N(v)| \geqslant \alpha + \omega - 1.$$

2. Let $K_1, K_2, \dots, K_\alpha$ be the partition of $G - v$ in α ω-cliques (we can suppose that $w \in K_1$) and let S be an α-stable containing w (so $S \subseteq N(v)$ and $\forall i$, $|S \cap K_i| = 1$). We can order vertices $w_1, \dots, w_\alpha$ in S such that (Lemma 46)

$$\mathcal{M}(w_1) \supseteq \mathcal{M}(w_2) \supseteq \cdots \supseteq \mathcal{M}(w_\alpha)$$

(we recall that $\mathcal{M}(w_i) = N(w_i) \cap M$). Now, suppose that there exist 3 ω-cliques intersecting $N(v)$ only on the vertices of S ($\alpha \geqslant 3$); then $\mathcal{M}(w_{i_1}) \subseteq \mathcal{M}(w_{i_2}) \subseteq \mathcal{M}(w_{i_3})$ and $\mathcal{M}(w_{i_1})$ contains an $(\omega - 1)$-clique. So $w_{i_1}w_{i_2}$, $w_{i_2}w_{i_3}$ and $w_{i_3}w_{i_1}$ are 3 co-critical pairs inducing a cycle, which contradicts Theorem 6. Then there exist at most two ω-cliques K_i and K_j intersecting $N(v)$ in only one vertex, so $d(v) \geqslant 2(\alpha - 1)$. $\square$

Proof of Theorem 19. Let G be a C_5-free partitionable graph and suppose that there exists a vertex $v \in V(G)$ (v not Int P_5) such that $d(v) = 2\omega - 2$. Let $S_1, S_2, \dots, S_\omega$ be the partition of $G - v$ in ω α-stable sets. According to Theorem 14 we have a partition of $G - v$ in $N(v)$ and $M = \mathcal{M}(w)$ ($w \in N(v)$). First, we can remark that $\forall i$ $(1 \leqslant i \leqslant \omega)$ $|S_i \cap N(v)| \geqslant 1$ (otherwise, $\{v\} \cup S_i$ would induce an $(\alpha + 1)$-stable set).

Claim 1. *There exist exactly α α-stable sets, from the partition $S_1, \dots, S_\omega$, intersecting $N(v)$ in one (and only one) vertex.*

First, we prove that there exist at least α such α-stable sets. We recall that there exist at least one α-stable set S_w such that $S_w \subseteq N(v)$, so there exist at most $2\omega - 2 - (\alpha + \omega - 1) + 1 = \omega - \alpha$ α-stable sets (including S_w) intersecting $N(v)$ in at least two vertices.

Moreover, there exist at most α such α-stable sets because if $|S_i \cap N(v)| = 1$ $\{v\} \cup (S_i \cap M)$ induces an α-stable set and v belongs to exactly α α-stable sets.

Now suppose that $S_1, \dots, S_\alpha$ are such that $|S_i \cap N(v)| = 1$ $S_{\alpha+1}, \dots, S_{\omega-k}$ are such that $|S_i \cap N(v)| \geqslant 2$ (and $S_i \cap M \neq \emptyset$) and $S_{\omega-k+1}, \dots, S_\omega \subseteq N(v)$ $(k \geqslant 1)$.

Claim 2. *If $k \geqslant 2$ then $G \simeq \overline{C_{2p+1}}$.*

We have

$$|N(v)| \geqslant k\alpha + \alpha + 2(\omega - k - \alpha),$$

$$2\omega - 2 \geqslant k\alpha + 2\omega - 2k - \alpha,$$

$$2(k - 1) \geqslant \alpha(k - 1).$$

We know that $k \geqslant 1$, so if $k \neq 1$ (that is if $k \geqslant 2$) we have $\alpha \leqslant 2$ and so $G \simeq \overline{C_{2p+1}}$.

From now on, we suppose that $k = 1$, we write $M_i = S_i \cap M(v)$, $N_i = S_i \cap N(v)$ (for $1 \leqslant i \leqslant \omega - 1$) and $N_i = \{s_i\}$ for $1 \leqslant i \leqslant \alpha$.

Claim 3. *The vertices s_i $(1 \leqslant i \leqslant \alpha)$ are adjacent to all the other vertices in $\bigcup_{1 \leqslant j \neq i \leqslant \omega - 1} N_j$.*

Assume not; there exists a vertex $x_j \in N_j$ ($j \neq i$ and $j \neq \omega$) such that $s_i x_j \notin E$. So, Lemma 48 implies that $\forall a \in M_i$, $\forall b \in M_j$, $ab \notin E$. Moreover, since $S_i \cup S_j$ induces a connected bipartite subgraph of G, there exists an induced path in $S_i \cup S_j$ between x_j and s_i. Let P be such a path and let m_i be one neighbour of x_j on P. One can remark that, for every vertex $b \in M_j$, we have $bs_i \notin E$ (otherwise $m_i x_j v s_i b$ induces a P_5 that contains v as an internal vertex). But in this case, b has no neighbours in S_i, a contradiction.

Now, let K_i be an ω-clique containing v and not s_i ($1 \leqslant i \leqslant \alpha$), we have $K_i - \{v\} \subseteq N(v)$ (because $v \in K_i$) so $K_i \cap S_w = \{w_i\}$ (because $K_i \cap S_i = \emptyset$), we can remark that $w_i s_i \notin E$ (otherwise, since s_i is adjacent to all $N(v) \setminus S_w$ we would obtain an $(\omega+1)$-clique) and that $w_i s_j \in E 1 \leqslant j \leqslant \alpha$, $j \neq i$ (because $K_i \cap S_j \neq \emptyset$).

This remark can be restated as follow, $\forall i$ ($1 \leqslant i \leqslant \alpha$) $s_i w_i \notin E$ and $\forall j \neq i s_i w_j \in E$.

Let i be an integer such that $w = w_i$, since w is in α α-stable sets of G, there exists an α-stable set S ($S \neq S_w$) containing $w_i = w$ (so $S \subseteq N(v)$). We have $S \cap K_i = \{w_i\}$, so, if $\alpha \geqslant 3$ we have $s_i \notin S$ (because s_i is adjacent to all $N(v) \setminus \{w_i\}$).

Now, let $Q = K_i - w_i + s_i$, Q is an ω-clique but $S \cap Q = \emptyset$ $S_w \cap Q = \emptyset$ and $S \neq S_w$, a contradiction. □

Proof of Theorem 20. Let G be a C_5-free minimal imperfect graph and let $v \in V(G)$ such that v is not Mid P_5. Moreover, we suppose that $N_G(v) = K \cup S$ induces a split graph (where K is a maximal clique and $S = \{s_1, s_2, \ldots, s_p\}$ ($p \leqslant \alpha$) a stable set).

We know ([22]) that in a minimal imperfect Berge graph, if there exists a vertex whose neighbourhood induces a split graph, then $N(v) = K \cup S$ has the following structure:

- K induces an $(\omega - 1)$-clique.
- S induces a stable set with at least $\omega - 1$ vertices, moreover, in S there exist $\omega - 1$ vertices s_i ($1 \leqslant i \leqslant \omega - 1$) that are in one-to-one correspondence with vertices in K such that each vertex s_i ($1 \leqslant i \leqslant \omega - 1$) is adjacent to all vertices in K except to its corresponding vertex (which is denoted by k_i).

Moreover, in this theorem, since G is C_5-free, we know that there exists an α-stable set $S_w \subseteq N(v)$ (cf. Theorem 14).

First, we claim that if $\omega \geqslant 4$ then for every α-stable set $S' \subseteq N(v)$ we have $S' \cap K = \emptyset$.

Indeed, assume that $S' \cap K \neq \emptyset$ (i.e. $S' \cap K = \{k\}$) then there exists i ($1 \leqslant i \leqslant \omega - 1$) such that $ks_i \notin E$ but $\forall j \neq i$ ($1 \leqslant j \leqslant \omega - 1$) $ks_j \in E$. Possibly S' also contains some vertices in $\{s_\omega, s_{\omega+1}, \ldots, s_p\}$ ($p \leqslant \alpha$). So,

$$|S'| \leqslant |\{k\}| + |\{s_i\}| + |\{s_\omega, \ldots, s_p\}| \quad (p \leqslant \alpha),$$

$$|S'| \leqslant 1 + 1 + (p - (\omega - 1)),$$

$$|S'| \leqslant 3 + \alpha - \omega.$$

But $|S'| = \alpha$, that is $\alpha \leqslant 3 + \alpha - \omega$, i.e. $\omega \leqslant 3$.

Now, if $\omega \leqslant 3$ then $G \simeq \overline{C_7}$ by Tucker's Theorem [37]. So we can suppose that $\omega \geqslant 4$, we know that vertex w is in α α-stable sets of G and that all these α-stable sets are included in $N(v)$. But Claim 3 implies that there exists at most one such α-stable set in $N(v)$ (if S is an α-stable set), a contradiction. $\square$

Proof of Theorem 24. We will use a technique similar to that used for proving the theorem of Chvátal and Erdős [10]. We suppose that G and H satisfy the hypothesis of the theorem. Since G is $2K_2$-free, for every minimal cutset C there exists a vertex v such that $C = N_G(v)$. We denote by $k(G)$ the connectivity of G, we have $k(G) = \min_{v \in V(G)} d(v)$. We recall that $\forall v \in V(G)\ d(v) \geqslant \alpha + \omega - 1$ (Theorem 17).

Let x, y be two vertices in H and let $H_1 = H - \{x, y\}$. The subgraph $G - H_1$ is connected since $k(G) = \min d(v) \geqslant \alpha + \omega - 1$; so, let us consider the longest path P between x and y in $G - H_1$. Assume, by contradiction, that P does not contain all the vertices in $G - H_1$ and let Q be a connected component of $G - (P + H_1)$. There exist at least $k(G)$ vertices in $P + H_1$ that have some neighbours in Q. Since $|H_1| \leqslant \omega - 2$, there exist at least $\alpha + 1$ such vertices in P (say $v_1, v_2, \ldots, v_{\alpha+1}$ when walking on P from x to y).

Any two vertices v_i and v_j are not consecutive on P; otherwise one can extend the path P with some vertices in Q. Let v_i^+ be the neighbour of v_i, between v_i and v_{i+1} on P. We claim that $\forall i, j\ (1 \leqslant i, j \leqslant \alpha,\ i \neq j)\ v_i^+ v_j^+ \notin E$, otherwise assume, for example, that $i < j$, then

$$x, \ldots, v_i,\ q_1, \ldots, q_p,\ v_j, \ldots, v_i^+,\ v_j^+, \ldots,\ v_{j+1}, \ldots, y,$$

where $q_1, \ldots, q_p$ are some vertices in $Q (p \geqslant 1)$, is a path longer than P. So $v_1^+, v_2^+, \ldots, v_\alpha^+$ induce an α-stable set of G, but since none of these vertices is adjacent to Q we obtain an $(\alpha + 1)$-stable set, a contradiction. $\square$

Proof of Theorem 25. Let G be a C_5-free minimal imperfect MP_5 graph, let $v \in V(G)$ (v not Mid P_5) and let $K_1, K_2, \ldots, K_\alpha$ be the partition of $G - v$ in α ω-cliques. We recall that there exists $w \in N(v)$ such that $M = \mathcal{M}(w)$. For every integer i $(1 \leqslant i \leqslant \alpha)$ we have $|K_i \cap N(v)| \geqslant 1$ (otherwise $\{w\} \cup K_i$ would induce an $(\omega + 1)$-clique) and $|K_i \cap M| \geqslant 1$ (otherwise $\{v\} \cup K_i$ would induce an $(\omega + 1)$-clique). Let us consider an α-stable set S containing w (so $S \subseteq N(v)$), we know that

$$\forall x, y \in N(v), \quad xy \notin E \Rightarrow \mathcal{M}(x) \subseteq \mathcal{M}(y) \quad \text{or} \quad \mathcal{M}(y) \subseteq \mathcal{M}(x).$$

So, we can order vertices $w_1, \ldots, w_\alpha$ in S such that

$$\mathcal{M}(w_1) \supseteq \mathcal{M}(w_2) \supseteq \cdots \supseteq \mathcal{M}(w_\alpha);$$

in particular, this implies that w_i is adjacent to all the vertices in $\mathcal{M}(w_j)$ for $j \geqslant i$. Moreover since M is connected, there exists an edge xy with $x \in M \cap K_1$ and $y \in M \setminus K_1$, say $y \in K_i \cap M$ $i \geqslant 2$. We write $\dot{K}_i = K_i \setminus \{w_i\}$, then

$$v, w_\alpha, \dot{K}_\alpha, w_{\alpha-1}, \dot{K}_{\alpha-1}, \ldots, w_i, \dot{K}_i \setminus \{y\}, y, x, \dot{K}_1 \setminus \{x\}, w_1, \dot{K}_2, w_2, \ldots, \dot{K}_{i-1}, w_{i-1}$$

is the required cycle. $\square$

Let us now recall some definitions introduced by Maffray and Preissmann in [21]. Let G be a minimal imperfect Berge graph, let G^{co} be the graph with the same vertex set as G and whose edges are the co-critical pairs of G and let V_2 be the set of vertices of G that belong to at least two co-critical pairs of G. We claim that $V \setminus V_2 \neq \emptyset$. Assume not, so there exists $v_2 \in V_2$ and then by Theorem 6 the connected component of G^{co} containing v_2 induces a tree. Since $v_2 \in V_2$, this tree contains some edges, so it must have a pendant vertex v_1 (i.e. a vertex of degree one in G^{co}) and $v_1 \notin V_2$, which contradicts $V \setminus V_2 = \emptyset$.

Lemma 49 (Maffray and Preissmann [21]). *Let $v \in V \setminus V_2$ and let $M = \mathcal{M}(w)$ be the set of all non-neighbours of v, then there exists a triangle in M.*

Corollary 50. *Let G be a P_5-free minimal imperfect Berge graph, then G contains a F_7.*

Proof. Let $v \in V \setminus V_2$, then there exists a triangle T in M (Lemma 49) and vwT induces an F_7.

Proof of Theorem 28. We will show that a P_5-free minimal imperfect Berge graph must contain a F_{10}. Let G be a P_5-free minimal imperfect graph and assume that G is F_{10}-free. Let $v \in V \setminus V_2$ and let $S_1, \dots, S_\omega$ be the partition of $G - v$ in ω α-stable sets. We will write $N_i = S_i \cap N(v)$ $M_i = S_i \cap M$ and $M_{123} = M_1 \cup M_2 \cup M_3$. Lemma 49 implies that there exist at least one triangle T included in M (say $T \subseteq M_{123}$) and Lemma 48 implies that N_1, N_2 and N_3 induce a complete tri-partite subgraph.

First we will study the structure of the subgraph induced by $S_1 \cup S_2 \cup S_3$. Let n_i be any vertex of N_i $(i = 1,2,3)$ and $\{t_i\} = T \cap M_i$ $(i = 1,2,3)$.

We know (Lemma 47) that $S_1 \cup S_2$ induces a connected (and P_5-free) bipartite subgraph so $n_1 t_2 \in E$ or $n_2 t_1 \in E$. First suppose that $n_2 t_1 \in E$, then $n_2 t_3 \notin E$ (otherwise $vn_2t_1t_3t_2$ induces an F_{10}).

Now, let us consider the bipartite graph induced by $S_2 \cup S_3$ $\forall x \in N_3$ we have $xt_2 \in E$ or $n_2 t_3 \in E$ and since $n_2 t_3 \notin E$ one has $\forall x \in N_3$, $xt_2 \in E$ and $\forall x \in N_3$, $xt_1 \notin E$ (otherwise $vxt_1t_2t_3$ induces an F_{10}).

Then, consider $S_1 \cup S_3$, we have $\forall y \in N_1$, $yt_3 \in E$ (or $n_3 t_1 \in E$ which is not possible) and $\forall y \in N_1$, $yt_2 \notin E$ (otherwise $vyt_2t_3t_1$ would induce an F_{10}).

Lastly, reconsider $S_1 \cup S_2$, we have $\forall z \in N_2$, $zt_1 \in E$ (or $n_1 t_2 \in E$ which is not possible) and $\forall z \in N_2$, $zt_3 \notin E$ (otherwise $vzt_1t_3t_2$ would induce an F_{10}). We can remark that we obtain the same structure if we choose $n_1 t_2 \in E$ instead of $n_2 t_1 \in E$.

Claim 4. $\forall u \in M_{123}$, $\exists T_u \subseteq M_{123}$ *such that* $u \in T_u$ *(where T_u is a triangle).*

Suppose that $u \in M_{123}$ is not in a triangle of M_{123}; we know that there exist $\omega\omega$-cliques in G containing u, let K be such an ω-clique. We can suppose that $u \in K \cap M_1$, so $K \cap N_1 = \emptyset$, moreover $K \cap N_2 \neq \emptyset$ or $K \cap N_3 \neq \emptyset$ (because K cannot intersect both M_2 and M_3). Let $T_u = \{m'_1, m'_2, m'_3\}$ be a triangle of M_{123} (Lemma 49).

First suppose that there exists $n_2 \in N_2$ such that $un_2 \in E$. We cannot have both $um'_2 \notin E$ and $um'_3 \notin E$ (otherwise $un_2n_1m'_3m'_2$ induces a P_5):

- if $um'_2 \in E$ then $um'_3 \notin E$ (otherwise u would be in a triangle $\{u, m'_2, m'_3\}$) and $um'_2m'_3n_1v$ induces a P_5;
- if $um'_3 \in E$ then $um'_2 \notin E$ and $vn_2um'_3m'_2$ induces a P_5.

So, there exists $n_3 \in N_3$ such that $un_3 \in E$ (and $un_2 \notin E$, $\forall n_2 \in N_2$). Remark that if $um'_3 \notin E$ then $un_3n_2m'_1m'_3$ induces a P_5 whereas if $um'_3 \in E$ $un_3n_2m'_1m'_3$ induces a C_5, a contradiction.

Claim 5. *Let K be an ω-clique of G, if $K \cap M_{123} \neq \emptyset$ then K intersect M_{123} in a triangle.*

Indeed, let $u \in K \cap M_{123}$, by Claim 4, u is in a triangle T_u of M_{123}, we will show that $K \cap (N_1 \cup N_2 \cup N_3) = \emptyset$. We can suppose that $u \in M_1 \cap T_u$ (say $T_u = \{u, u_2, u_3\}$) then $K \cap N_1 = \emptyset$ and since $u \in T_u$ either there exist no edge between u and N_2 or there exist no edge between u and N_3 (say between u and N_3, so $K \cap N_3 = \emptyset$). Let $b \in K \cap M_3$ and suppose that $K \cap N_2 \neq \emptyset$ (say $n_2 \in K \cap N_2$), since $n_2b \in E$ we have $u_2b \notin E$ (otherwise vn_2ubu_2 would induce an F_{10}) but then $u_3u_2n_3n_2b$ induces a P_5, a contradiction.

Claim 6. *There exists an α-stable set S_p ($p \neq 1,2,3$) such that $S_p \cap N(v) \neq \emptyset$ and $S_p \cap M \neq \emptyset$*

Indeed, suppose that $M = M_{123}$ and let us consider $K_1, \ldots, K_\alpha$ the partition of $G - v$ in α ω-cliques; we know that $\forall i$, $1 \leqslant i \leqslant \alpha$ $K_i \cap N(v) \neq \emptyset$ and $K_i \cap M \neq \emptyset$. So, Claim 5 implies that $\forall i$, $|K_i \cap M| = 3$ (because $M = M_{123}$), then $|M| = 3\alpha$. But we suppose that $M = M_{123}$ and we know that $\forall i$, $N_i \neq \emptyset$ which implies that $|M| \leqslant 3\alpha - 3$, a contradiction.

Now, let S_4 be an α-stable set such that there exists a vertex in M_4 which is adjacent to at least one vertex in M_{123} (such an α-stable set exists because M is connected and by Claim 6).

Claim 7. *No vertex in M_4 is adjacent to 3 vertices inducing a triangle in M_{123}.*

Assume not; let us consider the connected bipartite subgraph induced by $S_3 \cup S_4$ (remark that $N_1 \cup N_2 \cup N_3 \cup N_4$ induces a complete multi-partite graph by Lemma 48). Let $n_4 \in N_4$ and $m_4 \in M_4$ such that there exists a triangle $T_{123} \subseteq N_{M_{123}}(m_4)$ (say $T_{123} = \{m_1, m_2, m_3\}$). We have $m_3n_4 \in E$ (or $n_3m_4 \in E$ but in this case, $vn_3m_2m_4m_3$ induces a F_{10}), we also have $n_2m_4 \in E$ (consider $S_2 \cup S_4$). Now let us consider the triangle included in $M_1 \cup M_3 \cup M_4$ and the bipartite subgraph induced by $S_1 \cup S_4$. We have $n_1m_4 \in E$ or $n_4m_1 \in E$ but in both cases we obtain an induced F_{10}.

Claim 8. *No vertex m_4 in M_4 is adjacent to exactly one vertex of a triangle $\{m_1, m_2, m_3\}$ in M_{123}.*

Assume not and suppose that $m_3m_4 \in E$ then $m_4n_3 \in E$ (otherwise vertices $m_4m_3m_2n_3v$ induces a P_5) and $m_4n_2 \in E$ (otherwise $m_4m_3m_1n_2n_3$ induces a C_5) but then $m_3m_4n_2n_3v$ induces a F_{10}.

We can now end the proof, let $T_1 = \{m_1, m_2, m_3\}$ be a triangle in M_{123} such that there exists a vertex $m_4 \in M_4$ adjacent to exactly two vertices (say m_2 and m_3) of T_1 (such a triangle exists by the choice of S_4 and Claims 4, 7 and 8) and let $T_2 = \{m_2, m_3, m_4\}$.

Let K be an ω-clique containing $u = m_2 \in T_1 \cap T_2$, Claim 5 implies that $|K \cap M_{123}| = 3$ (say $K \cap M_{123} = T_K$), so $K \cap M_4 = \emptyset$ (Claim 7). But K must intersect each α-stable set, then $K \cap N_4 \neq \emptyset$. Let $n_4 \in K \cap N_4$, since $m_2 \in T_K$ and $m_2m_4 \in E$, there exists $t \in T_K$ ($t \neq m_2$) such that $tm_4 \in E$ (Claim 8) and then $vn_4m_2tm_4$ induces an F_{10}, a contradiction. $\square$

Proof of Theorem 30. We will prove a stronger statement.

Theorem 51. *If a Berge graph contains no P_5 and no F_{12} then either it is the complement of a bipartite graph or it contains a star-cutset.*

Proof. Our proof is similar to that used in [18] to short prove a result of Olariu [28]. We consider a Berge graph G such that neither $\overline{G}$ induces a bipartite graph, nor G contains a star-cutset and we will show that there must exist a P_5 or an F_{12}.

Since G is a Berge graph and $\overline{G}$ does not induce a bipartite graph, we know that there exists a stable set of size three in G (say $\{u_1, u_2, u_3\}$). Let $Y = V - (N_G(u_1) \cup N_G(u_2) \cup \{u_1, u_2\})$; note that $Y \neq \emptyset$ because the vertex u_3 belongs to one connected component Y' of Y. Let A (resp. B) be the subset of vertices in $N_G(u_1)$ (resp. $N_G(u_2)$) that are adjacent to at least one vertex in Y'.

Claim 9. *$A' = A - N(u_2)$ and $B' = B - N(u_1)$ are not empty.*

Otherwise $\{u_2\} \cup N(u_2)$ would induce a star disconnecting u_1 and Y' in G, a contradiction.

Claim 10. $\forall a \in A'$, $\forall b \in B'$, $ab \in E$.

Otherwise, let P be a shortest path between a and b such that all interior vertices of P are in Y' (at least one); then u_1Pu_2 induces a P_5, a contradiction.

Claim 11. $\forall a \in A'$, $\forall x \in A' \cap B'$, $ax \in E$.

Otherwise, let P be a shortest path between x and a such that all interior vertices of P are in Y'.

- If P is of length 2 then let t be the only vertex of P in Y', u_2xu_1ta induces an F_{12}, a contradiction.
- If P is of length at least 3, let P be $xt_1P't_2a$ (with $|P'| \geqslant 0$) then $t_1xu_1at_2$ induces a P_5 or a C_5, a contradiction.

Claim 12. $\forall a_1, a_2 \in A'$, $a_1 \neq a_2$, $a_1a_2 \in E$.

Otherwise, let $a_1, a_2 \in A'$ such that $a_1a_2 \notin E$ and let $b' \in B' \neq \emptyset$ (Claim 9) then $u_2b'a_1a_2u_1$ induces an F_{12}, a contradiction.

Now let $a' \in A'$, by Claims 10, 11 and 12, we know that a' is adjacent to all the other vertices in $A \cup B$ (since $A \cup B \cap (N(u_1) \cap N(u_2)) = A \cap B$) and so $A \cup B$ induces a star disconnecting u_1 and Y', a contradiction. □

Proof of Theorem 32. The first part of this proof is similar to that used in [18]. Let G be a (P_5, F_{14})-free minimal imperfect Berge graph, we may assume that $\overline{G}$ contains a diamond D [38] and that neither $G = (V, E)$ nor $\overline{G}$ contains a star-cutset.

Let u_1, u_2, u_3, u_4 be the vertices of D such that u_3u_4 is the only edge of $\overline{D}$. Let $Y = V - (N_G(u_1) \cup N_G(u_2) \cup \{u_1, u_2\})$, note that $Y \neq \emptyset$ because the edge u_3u_4 belongs to one connected component Y' of Y. Let A (resp. B) be the subset of vertices in $N_G(u_1)$ (resp. $N_G(u_2)$) that are adjacent to at least one vertex in Y'.

Let us consider the vertex u_1, since G is P_5-free one has a partition of $G - u_1$ as in Theorem 14. In particular, there exists a vertex $w \in N(u_1)$ such that $M = \mathcal{M}(w)$ (where $M = V(G) \setminus (\{u_1\} \cup N(u_1))$). It is easy to see that w must be in $A \cap B$ (since w is adjacent to both vertex u_2 and subset Y') but then, $u_1wu_2u_3u_4$ induces an F_{14}, a contradiction. □

Proof of Theorem 33. Let G be a P_5-free minimal imperfect Berge graph and suppose that G is $(K_5 - e)$-free. Let v be a vertex of maximum degree in G, we know that $G - v$ has a (unique) partition in α ω-cliques. Let K be any ω-clique of this partition (we recall that G has the structure described in Theorem 14).

- K is not in $N(v)$, otherwise $K + v$ would induce an $(\omega + 1)$-clique.
- K is not in $M = \mathcal{M}(w)$, otherwise $K + w$ would induce an $(\omega + 1)$-clique.

If $|K \cap N(v)| \geqslant 3$, since $K \cap M \neq \emptyset$ there exist $x \in M \cap K$ and three vertices a, b and c in $N(v) \cap K \cap N(x)$ such that $\{v, a, b, c, x\}$ induces a $K_5 - e$, a contradiction.

So, $|K \cap N(v)| \leqslant 2$ and $|K \cap M| \geqslant \omega - 2$. Let $K_1, K_2, \ldots, K_\alpha$ be the α ω-cliques of the partition of $G - v$. We have

$$\forall i \quad (1 \leqslant i \leqslant \alpha) \quad |K_i \cap N(v)| \leqslant 2 \quad \text{and} \quad |K_i \cap M| \geqslant \omega - 2.$$

So, $d(v) \leqslant 2\alpha$ and $d(w) \geqslant \alpha(\omega - 2) + 1$. Since v is a vertex of maximum degree, we must have

$$2\alpha \geqslant \alpha\omega - 2\alpha + 1,$$

$$\alpha(\omega - 4) + 1 \leqslant 0.$$

But $\omega \geqslant 4$ by Tucker's theorem [37], a contradiction. □

Proof of Theorem 34. Let G be a MP_5 minimal imperfect Berge graph. Let v be a vertex of G (v not Mid P_5) and let w and $M = \mathcal{M}(w)$ be as in Theorem 14.

Let us consider an α-stable set S containing w, so $S \subseteq N(v)$ and Lemma 46 enables us to order vertices in S such that, for $1 \leqslant i,\ j \leqslant \alpha$

$$\mathscr{M}(s_i) \subseteq \mathscr{M}(s_j) \quad \text{iff } i \geqslant j$$

since $N(v)$ is a minimal cutset, we have $\mathscr{M}(s_\alpha) \neq \emptyset$, then for $t \in \mathscr{M}(s_\alpha)$ the subset $\{v,t\} \cup S$ induces a $K_{2,\alpha}$. $\square$

Proof of Theorem 37. Let G be a minimal imperfect IP_5 Berge graph; let v be a vertex such that v is not Int P_5 and let S_1, S_2, S_3, S_4 be the partition of $G - v$ in ω α-stable sets (we can suppose that $\omega \geqslant 4$ by Tucker's theorem).

We know that there exists at least one α-stable set S_1 included in $N(v)$ (see Theorem 14) moreover, since M is connected, there exist at least two α-stable sets S_i and S_j such that $S_i \cap N(v) \neq \emptyset$ and $S_i \cap M \neq \emptyset$. So, we have to study the following two cases:

Case 1. $S_i \cap M \neq \emptyset$ $(i = 2,3,4)$: Since M is connected, one can suppose that there exist an edge between $S_2 \cap M$ and $S_3 \cap M$ and an edge between $S_3 \cap M$ and $S_4 \cap M$ (with a relabeling of the stable sets, if necessary). Let $N_i = S_i \cap N(v)$, by Lemma 48 we know that $N_2 \cup N_3$ and $N_3 \cup N_4$ induce a complete bipartite graph. Even if there is no edges between $S_2 \cap M$ and $S_4 \cap M$, we know that there exists at least one edge between N_2 and N_4 (because $S_2 \cup S_4$ is connected).

1.1. *There exist at least two edges xy and $x'y'$ between N_2 and N_4 (possibly $x = x'$ or $y = y'$).* These two edges together with a vertex in N_3 induce at least two distinct triangles in $N(v)$ and these two triangles together with v induce two 4-cliques not intersecting S_1, a contradiction.

1.2. *There exists only one edge xy between N_2 and N_4.* First, we can remark that, in this case, we have $|N_2| = |N_4| = 1$ because $S_2 \cup S_4$ must induce a connected bipartite graph and there is no edge between $S_2 \cap M$ and $S_4 \cap M$. Moreover, $|N_3| = 1$, otherwise there exist two triangles in $N(v)$ not intersecting S_1, a contradiction. Therefore, in this case, $N(v)$ induces a split graph which contradicts Theorem 20.

Case 2. $S_i \subseteq N(v)$ $(i = 1,2)$ and $S_j \cap M \neq \emptyset$ $(j = 3,4)$: Since the vertex v is in four 4-cliques, there exist 4 triangles in $N(v)$ and $N_3 \cup N_4$ is a transversal of these triangles (because $N(v) - N_3 \cup N_4 = S_1 \cup S_2$ induces a bipartite graph). Moreover, it is a minimum transversal, indeed suppose that T induces a transversal such that $|T| < |N_3 \cup N_4|$, then $N(v) - T$ is bipartite and contains at least $2\alpha + 1$ vertices, therefore, we can find an $(\alpha + 1)$-stable set, a contradiction. These four triangles included in $N(v)$ are

$$s_1 s_2 n_3,\ s_1' s_2' n_4',\ s_1'' n_3'' n_4'',\ s_2''' n_3''' n_4''',$$

where s_i, s_i', s_i'', s_i''' (resp. n_j, n_j', n_j'', n_j''') are some vertices in S_i (resp. N_j). Since $N_3 \cup N_4$ is a minimum transversal of these 4 triangles, we have $|N_3 \cup N_4| \leqslant 4$ and we know that $|N_3 \cup N_4| \geqslant 2$ (otherwise, $S_3 + v$ or $S_4 + v$ would induce an $(\alpha + 1)$-stable set). So, we have only three cases to study:

2.1. $|N_3 \cup N_4| = 4$. Since $N_3 \cup N_4$ is a minimum transversal of the four triangles, they are vertex-disjoints and then $|N_3| \geqslant 3$ $|N_4| \geqslant 3$, so $|N_3 \cup N_4| \geqslant 6$, a contradiction.

2.2. $|N_3 \cup N_4| = 3$. We can suppose, w.l.o.g., that $|N_3| = 2$ and $|N_4| = 1$ (say $N_4 = \{n_4\}$), then the three triangles intersecting N_4 contain n_4 and, therefore, the vertex n_4 together with any vertex in the triangle not intersecting N_4 would induce a transversal smaller than $N_3 \cup N_4$, a contradiction.

2.3. $|N_3 \cup N_4| = 2$. Let $N_3 = \{p\}$ and $N_4 = \{q\}$ and remark that vp and vq are two critical edges. Now, let us consider a K_4 containing p and not v, this K_4 must intersect S_1, S_2 in a, b and S_4 in a vertex q'. Let us remark that $q \neq q'$ (otherwise $\omega = 5$) and then $q'v$ is a co-critical pair. By considering a K_4 that contains q and not v, we can find a vertex $p' \in S_3 \cap M$ that form a co-critical pair with v. But $\{v, p, q, p', q'\}$ induces a configuration which contradicts Theorem 7. □

Proof of Theorem 39. Let G be a minimal imperfect IP_5 Berge graph, let v be a vertex which is not Int P_5 and let $K_1, \ldots, K_\alpha$ be the partition of $G - v$ in α ω-cliques.

Suppose that there exists an integer i $(1 \leqslant i \leqslant \alpha)$ such that $|K_i \cap N(v)| = 1$, then $\omega(M) = \omega - 1$. Let us consider $S_1, \ldots, S_\omega$ the partition of $G - v$ in ω α-stable sets, suppose that $w \in S_1$; then $\forall i$ $(2 \leqslant i \leqslant \omega)$ $M_i = S_i \cap M \neq \emptyset$ (because $\omega(M) = \omega - 1$). So $N_2, \ldots, N_\omega$ (where $N_i = S_i \cap N(v)$) induce a complete multi-partite graph (Lemma 48), then $\forall i$ $(2 \leqslant i \leqslant \omega)$ $|N_i| = 1$ (because there exists exactly one ω-clique not intersecting S_1) and then $N(v)$ induces a split graph, a contradiction (Theorem 20). So, we have $\forall i$, $|K_i \cap N(v)| \geqslant 2$.

If there exists an integer i such that $|K_i \cap M| \geqslant 2$ then we obtain a F_4 (since $|K_i \cap N(v)| \geqslant 2$), a contradiction. So we can suppose that $\forall i$, $|K_i \cap M| = 1$. An α-stable set containing v must intersect M in $\alpha - 1$ vertices, then, since M is connected, we obtain that M is isomorphic to $K_{1,\alpha-1}$. But, there exists α α-stable sets containing v, a contradiction. □

Proof of Theorem 41. Let G be a P_5-free minimal imperfect Berge graph, let $v \notin V_2$ (i.e. a vertex that belongs to at most one co-critical pair of G) and let $K_1, \ldots, K_\alpha$ be the partition of $G - v$ in α ω-cliques. We know that $\forall i$ $(1 \leqslant i \leqslant \alpha)$ $K_i \cap N(v) \neq \emptyset$ and $K_i \cap M \neq \emptyset$. Let S be an α-stable set containing w (so $S \subseteq N(v)$ and $\forall i$ $|S \cap K_i| = 1$). We can order vertices $w_1, \ldots, w_\alpha$ in S such that

$$\mathcal{M}(w_1) \supseteq \mathcal{M}(w_2) \supseteq \cdots \supseteq \mathcal{M}(w_\alpha).$$

Moreover, since $v \notin V_2$, there exists at most one integer i such that $|K_i \cap M| = 1$. So, at least one of $\mathcal{M}(w_{\alpha-1})$ and $\mathcal{M}(w_\alpha)$ contains an edge which will form a complete join with $\{w_1, \ldots, w_{\alpha-1}\}$. □

Proof of Theorem 44. Let G be a minimal imperfect Berge graph with no induced F_6, let v be a vertex of G which is not Mid P_5 and let $K_1, \ldots, K_\alpha$ be the partition of $G - v$ in α ω-cliques. Let $w \in N(v)$ and $M = \mathcal{M}(w)$ be as in Theorem 14. We know that all α-stable sets containing w are included in $N(v)$ and that w belongs to exactly α α-stable sets pairwise differents in at least one vertex. Moreover such an α-stable set contains one (and exactly one) vertex from each K_i ($1 \leqslant i \leqslant \alpha$) and two differents α-stable sets must differ from at least one vertex. So, since $\alpha > 2$, there exists an integer i ($1 \leqslant i \leqslant \alpha$) such that $|K_i \cap M| \geqslant 1$ and w is not adjacent to an edge e in $K_i \cap N(v)$. Then $u \in K_i \cap M, w$ and v together with e induce an F_6, a contradiction. □

Proof of Theorem 45. Let G be a P_5-free minimal imperfect Berge graph and suppose that G is F_3-free. Let v be a vertex of maximum degree, let $S_1, \dots, S_\omega$ be the partition of $G - v$ in ω α-stable sets, and suppose that $w \in S_1$.

Since $S_1 \cup S_2$ induces a P_5-free connected subgraph (Lemma 47) we can order vertices $s_1^1, s_2^1, \dots, s_\alpha^1$ in S_1 (resp. $s_1^2, s_2^2, \dots, s_\alpha^2$ in S_2) in such a way that

$$N_{S_2}(s_i^1) \subseteq N_{S_2}(s_j^1)$$

(resp. $N_{S_1}(s_i^2) \subseteq N_{S_1}(s_j^2)$) whenever $i \geqslant j$. Note that $N(v)$ contains no C_4 and let $N_i = S_i \cap N(v)$ for $2 \leqslant i \leqslant \omega$ (possibly $N_i = S_i$).

We claim that $\forall i$ ($2 \leqslant i \leqslant \omega$) $|N_i| \leqslant 2$. Otherwise, suppose that there exists an integer i such that $|N_i| \geqslant 3$ (say i=2), and suppose that $N_2 = \{s_{n_1}^2, s_{n_2}^2, \dots, s_{n_p}^2\} \subseteq \{s_1^2, s_2^2, \dots, s_\alpha^2\}$ (with $N_{S_1}(s_{n_1}^2) \supseteq N_{S_1}(s_{n_2}^2) \supseteq \dots \supseteq N_{S_1}(s_{n_p}^2)$) and that $N_{S_1}(N_2) = \{s_{m_1}^1, s_{m_2}^1, \dots, s_{m_q}^1\}$. We have $s_{n_1}^2 s_{m_1}^1 \in E$ but, since $N(v)$ is C_4-free, we cannot have any edge $s_{n_i}^2 s_{m_j}^1$ between N_2 and $N_{S_1}(N_2)$ with $n_i \neq n_1$ and $m_j \neq m_1$. Then, as soon as $|N_2| \geqslant 3$, the subset $(S_1 \setminus \{s_{m_1}^1\}) \cup (N_2 \setminus \{s_{n_1}^2\})$ induces a stable set containing at least $\alpha + 1$ vertices. Therefore $\forall i \geqslant 2$, we must have $|N_i| \leqslant 2$.

Moreover, if S_i, S_j ($i, j \geqslant 2$) are two α-stable sets satisfying $\exists m_i \in S_i \cap M$, $\exists m_j \in S_j \cap M$ such that $m_i m_j \in E$ then $|N_i| = 1$ or $|N_j| = 1$ since there exists a complete join between N_i and N_j (Lemma 48).

We now claim that there exist at least two integers i and j such that $|N_i| = |N_j| = 1$. Otherwise, suppose that there exists only one integer i such that $|N_i| = 1$, the preceding fact implies that $\forall j$, $\forall k$, ($2 \leqslant j, k \leqslant \omega$), with $j \neq i$ and $k \neq i$ there exists no edge between M_j and M_k. So, as soon as we have 3 such α-stable sets (i.e. as soon as $\omega \geqslant 5$) one must have $3(\alpha - 2) \leqslant \alpha$, that is $\alpha \leqslant 3$, a contradiction. This implies that there exist at least two integers i and j such that $|N_i| = |N_j| = 1$. So

$$\begin{aligned} d(v) &\leqslant |S_1| + \sum_{p \neq i,j} |N_p| + |N_i| + |N_j| \\ &\leqslant \alpha + 2(\omega - 3) + 2 \end{aligned}$$

and

$$\begin{aligned} d(w) &\geqslant \sum_{p \neq i,j} |S_p \setminus N_p| + |S_i \setminus N_i| + |S_j \setminus N_j| + 1 \\ &\geqslant (\alpha - 2)(\omega - 3) + \alpha - 1 + \alpha - 1 + 1. \end{aligned}$$

Since the vertex v has been chosen of maximum degree, we must have $d(v) \geqslant d(w)$, that is $4(\omega - 2) - 1 \geqslant \alpha(\omega - 2)$ and since $\omega \geqslant 3$ we obtain

$$\alpha \leqslant 4 - \frac{1}{\omega - 2} \quad \text{i.e.} \quad \alpha \leqslant 3$$

a contradiction. □

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